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Two Dimensional Gravity with Boundary

Parthasarathi Majumdar[1]

Fermi National Accelerator Laboratory
P. O. Box 500, Batavia, IL 60510
and
Department of Physics and Astronomy
University of Maryland, College Park, MD 20742

ABSTRACT

An attempt is made to incorporate the effects of a boundary in the conformal gauge solution of two dimensional gravity. We discuss some possible choices for boundary conditions on the Liouville field and their implications for the renormalization of the central charge.

[1]Address after August 1, 1991 : Institute of Mathematical Sciences, CIT Campus, Madras 600 113, India.

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1. Introduction

Several years ago, Alvarez [1] made an in-depth analysis of the corrections to the Polyakov string due to the presence of boundaries and handles on the two dimensional world sheet. The main motivation was to study the Polyakov string [2] as a phenomenological model of Wilson loops in quantum chromodynamics. Despite the elegance and comprehensive nature of this analysis, a complete solution of the problem proved elusive largely because of the absence of a framework to probe the physics of the Liouville mode. Over the last few years, the latter has been investigated extensively, both in the continuum where the renormalization of all couplings and anomalous dimensions of a class of vertex operators have been obtained [2,3], and also through discretization of the random surface which is then mapped to certain large order matrix models leading to exact solutions for conformal matter couplings with central charge $c \leq 1$ [4]. However, with some notable exceptions [5,6], the effects of the presence of a boundary on the 2d random surface have been ignored. My purpose in this note is to address this issue in the case of continuum 2d gravity using the approach of ref. [3], supplemented by the works of Mavromatos and Miramontes [7] and D'Hoker and Kurzepa [8]. In particular, we re-examine the analysis in ref. [1] to see if an improved insight can indeed be obtained. Although this problem clearly pertains to the *open* rather than the closed string, boundary effects may play an important role if the low lying excitations of QCD eventually turn out to be describable in terms of a string theory. This is true even for the effective string theory currently being considered as a model of hadrons [9], if one wishes to include chiral fermions as sources of chromodynamic flux tubes.

This work is of a preliminary nature; we begin by briefly reviewing the arguments of Alvarez [1] for his choice of boundary conditions. This is followed by a discussion of the functional measures needed to define the partition function of the theory. The functional measure for the Liouville field is chosen to be invariant under a Weyl transformation of the fiducial metric and a simultaneous translation of the Liouville field. We then follow ref.s [7] and [8] to compute the Jacobian emerging from a transformation of the functional measure to one that is translationally invariant. This enables us to obtain the renormalizations of the coupling constants and anomalous dimensions of certain operators. We find that, with the choice of boundary conditions espoused in [1], boundary effects do not alter the analysis in the bulk [3] in so far as the central charge and the renormalization of the cosmological constant operator are concerned. We also show that because the boundary condition proposed in [1] relates the Liouville field to reparametrizations of the affine parameter on the boundary, integrating over all such reparametrizations leads to an additional boundary term in the Liouville action, while the Liouville field itself is rendered free on the boundary. Furthermore, the 'naive' energy momentum tensor calculated from this action has contributions from the boundary which make the trace nonvanishing even when the cosmological terms are renormalized away. The vanishing of these terms entails extra constraints on the boundary value of the Liouville field which are very similar to the Neumann boundary conditions used by Martinec et. al. [5] to study boundary operators in 2d gravity.

2. Boundary Conditions

Following [1] we consider a compact two-fold M with boundary ∂M, with local coordinates z^a , $a = 1, 2$, and metric $g_{ab}(z)$, assumed to have an Euclidean signature. The boundary is parametrized by the real affine parameter s, and defined by the curve $z_0^a \; = \; z_0^a(s)$. The line element on the boundary is given by

$$dS^2 \; = \; g_{ab}(z_0(s))dz_0^a dz_0^b \; = \; g_{ab}(s)\dot{z}_0^a \dot{z}_0^b ds^2 \quad , \tag{2.1}$$

where, $\dot{z}_0 \; \equiv \; dz_0(s)/ds$; thus, defining the frame field $E(s)$ on the boundary as

$$E(s) \; \equiv \; \left[g_{ab}(z_0(s))\dot{z}_0^a \dot{z}_0^b\right]^{\frac{1}{2}} \quad ,$$

the line element may be reexpressed as $dS \; = \; E(s)ds$. The unit tangent and normal vectors to the boundary are then defined as

$$t^a \; \equiv \; \dot{z}_0^a/(\dot{z}_0^2)^{\frac{1}{2}} \; = \; E^{-1}\dot{z}_0^a \quad ,$$

and

$$n^a \; \equiv \; \epsilon^{ab} t_b \; = \; \epsilon^{ab} E^{-1} \dot{z}_{0,b} \quad .$$

In the Polyakov formulation [2] of the bosonic string, the integration in the partition function over the two dimensional intrinsic metric g_{ab} can be factored after a conformal gauge fixing ($g_{ab} \; = \; e^{\phi}\hat{g}_{ab}$) into integrations over infinitesimal diffeomorphisms $\omega^a(z)$ and the Liouville field $\phi(z)$. For surfaces without boundary, this yields a Jacobian which is independent of the the $\omega^a(z)$, as is the action through its reparametrization invariance. Thus the integral over the diffeomorphisms is eliminated by dividing the functional integral by the volume of the group of infinitesimal diffeomorphisms. When the world sheet has a boundary, the question of the boundary condition on ω needs to be addressed. The naive choice $\omega^a|_{\partial M} = 0$ contradicts the fact that the ω^a must satisfy the equation [2]

$$\nabla_{(a}\omega_{b)} \; - \; \tfrac{1}{2} g_{ab} \nabla^c \omega_c \; = \; \gamma_{ab} \quad , \tag{2.2}$$

where, γ_{ab} is a traceless symmetric second rank tensor field which does not necessarily vanish on the boundary $z = z_0(s)$.[2] One needs to allow for reparametrizations of the affine parameter s parametrizing the boundary $s \rightarrow \alpha(s)$. Thus the preferred boundary condition on the infinitesimal diffeomorphisms requires [1,2] that they vanish on the boundary *modulo* infinitesimal reparametrizations of the affine parameter. In order that s-reparametrizations do not affect the shape of the boundary, one imposes the further condition [1,2]

$$n^a \omega_a(z_0(s)) \;=\; 0 \;\; , \tag{2.3}$$

so that

$$\omega^a(z_0(s)) \;=\; t^a \delta\alpha(s) \;\; , \tag{2.4}$$

where $\delta\alpha(s)$ corresponds to an infinitesimal reparametrization of the affine parameter s.

The boundary condition (2.4) has the immediate consequence [1] that

$$n^a t^b \left[\nabla_{(a}\omega_{b)} \;-\; \tfrac{1}{2} g_{ab} \nabla^c \omega_c\right] \;=\; k\alpha(s) \tag{2.5}$$

on the boundary ∂M. This is derived using the following geometrical relation,

$$t^a \nabla_a t^b \;=\; k(s) n^b \;\; , \tag{2.6}$$

where $k(s)$ is the geodesic (extrinsic) curvature on the boundary. This quantity also appears in the formula for the Euler characteristic of a two dimensional Riemannian manifold with a boundary,

$$2\pi\chi(M) \;\;=\;\; \tfrac{1}{2}\int_M \sqrt{g} R(g) \;+\; \int_{\partial M} Ek \;\; ,$$

and has the property $k \;=\; k_0 \;\neq\; 0 \; for \; g_{ab} \;=\; \eta_{ab}$.

At point one must also specify the boundary condition on the Liouville field ϕ in order that the partition function may be evaluated. In ref. [1], the two-fold is embedded in spacetime in a manner such that the boundary ∂M coincides with a curve C which can be identified with a Wilson loop. In doing so, Alvarez chooses the modified Dirichlet boundary condition

$$h_{ab} n^a t^b \;=\; 0 \;\; on \; \partial M \;\; , \tag{2.7}$$

where h_{ab} is the *induced* metric on the world sheet. Imposing the restriction that the length of the boundary as measured by the induced metric must be identical with that

[2] Eq. (2.2) follows from the attempt to gauge fix the metric to the conformal gauge by infinitesimal diffeomorphisms.

measured by the intrinsic metric, he obtains the following restriction on infinitesimal deformations of ϕ at ∂M

$$\delta\phi(z_0(s)) \ + \ t^a t^b \nabla_a \omega_b \ = \ 0 \ \ on \ \partial M \ \ . \qquad (2.8)$$

Using eq.s (2.4) and (2.6), this equation simply implies that

$$\delta\phi(z_0(s)) \ + \ 2E^{-1}\frac{d}{ds}\delta\alpha(s) \ = \ 0 \ \ on \ \partial M \ \ . \qquad (2.9)$$

This last relation will be of importance in the sequel. We remark that this is also a modified Dirichlet boundary condition on the Liouville field ϕ.

3. Functional Measures

As stated earlier, the passage to the conformal gauge

$$g_{ab} \ = \ e^{2\phi}\hat{g}_{ab} \ , \qquad (3.1)$$

where $\hat{g}_{ab}$ is the fiducial 2-metric, implies that $\int \mathcal{D}g \ \rightarrow \ \int \mathcal{D}\omega\mathcal{D}\phi\mathcal{J}$, with $\mathcal{J}$ the Jacobian of the transformation of integration variables. Further, the integration over the infinitesimal diffeomorphism ω is given by eq. (2.4) on the boundary. Now, observe that as far as the *bulk* diffeomorphism is concerned, the Jacobian in (3.1) as well as the action are independent of it, so that, upon division of the functional integral by the volume of the group of infinitesimal diffeomorphisms in the bulk, one obtains ,

$$\int \frac{\mathcal{D}\omega}{\mathcal{V}_{diff}}\mathcal{D}\phi \ = \ \int \mathcal{D}\alpha\mathcal{D}\phi\tilde{\mathcal{J}} \ \ . \qquad (3.2)$$

The integral over the reparametrizations $\alpha(s)$ of the boundary ∂M is, however, non-trivial since the boundary condition on the Liouville field depends on $\alpha(s)$.

Thus, the conformal (ϕ-dependent) part of the Polyakov functional integral, after evaluation of the functional determinants arising from gauge fixing [1] takes the form (with g_{ab} the fiducial metric from now on)

$$Z(g,E) \ = \ \int \mathcal{D}\phi\mathcal{D}\alpha e^{-S(g,E,\phi)}\delta[\phi \ + \ 2E^{-1}\dot{\alpha}(s)] \ \ , \qquad (3.3)$$

where

$$\begin{aligned} S(g,E,\phi) = \ & S(g,E,0) \ + \ Q_0^2\left[I_v(g,\phi) \ + \ I_b(E,\phi)\right] \ + \ \mu_0^2 A(g,\phi) \\ & + \ \lambda_0 B(E,\phi) \ + \ \rho_0 F(E,\phi) \ \ , \end{aligned} \qquad (3.4)$$

with

$$
\begin{aligned}
I_v(g,\phi) &\equiv \frac{1}{4\pi}\int_M d^2z\sqrt{g}\,\left[\tfrac{1}{2}g^{ab}\partial_a\phi\partial_b\phi \;+\; \tfrac{1}{2}R(g)\phi\right] \\
I_b(E,\phi) &\equiv \frac{1}{4\pi}\int_{\partial M} Ek\phi \\
A(g,\phi) &\equiv \frac{1}{4\pi}\int_M d^2z\sqrt{g}e^{2\phi} \\
B(E,\phi) &\equiv \frac{1}{8\sqrt{\pi}}\int_{\partial M} dsEe^{\phi} \\
F(E,\phi) &\equiv \frac{1}{8\pi}\int_{\partial M} dsEn^a\partial_a\phi \quad .
\end{aligned}
\tag{3.5}
$$

Here the coupling constant $Q_0^2 = \frac{1}{3}(26 - c_m)$, and the other 'bare' parameters are free.[3] Throughout (3.4-5) and the sequel, g_{ab} will denote the *fiducial* metric on the 2-fold and E the corresponding einbein (frame field) on ∂M. The delta functional in (3.3) constrains the ϕ field on the boundary to obey the boundary condition (2.9). The presence of this delta functional makes the integral over the $\alpha(s)$ non-trivial, unlike the integral over the infinitesimal diffeomorphisms ω.

The functional measure for the Liouville field is defined through the metric on the space of functions $\{\phi\}$ [1,2]

$$||\delta\phi||_W^2 \;=\; \int_M d^2z\sqrt{g}e^{2\phi}(\delta\phi)^2. \tag{3.6}$$

Clearly, this metric and the measure defined from it are invariant under the Weyl transformations $g_{ab} \rightarrow e^{2\sigma}g_{ab}$, $\phi \rightarrow \phi - \sigma$. The transition to a measure that is invariant under translations of ϕ alone, namely

$$||\delta\phi||_t^2 \;=\; \int_M d^2z\sqrt{g}(\delta\phi)^2 \tag{3.7}$$

will entail a Jacobian [3]. This has been calculated explicitly in ref.s [7] and [8], and found to be identical in form to the Liouville action, for the case when the twofold has no boundary, i.e., the action is the sum of the terms I_v and A in (3.4). In our case there is also the additional integral over the boundary reparametrizations $\alpha(s)$ whose measure may be chosen to be invariant under translations of the Liouville field :

$$||\delta\alpha(s)||_t^2 \;=\; \int_{\partial M} ds\; E(\delta\alpha)^2 \;\; ; \tag{3.8}$$

[3] Although for the case at hand, c_m denotes the spacetime dimensionality, it could be taken to indicate the central charge of any conformal (matter) field theory coupled to gravity.

thus, this choice ensures that the only contributions to the Jacobian come from the change in the measure for the Liouville field. If we had alternatively chosen a *Weyl*-invariant measure for $\alpha(s)$, additional boundary contributions would indeed have had to be taken into account. We shall comment on these later.

It is convenient to rewrite the functional integral (3.3) as

$$Z(g,E) \;=\; \int \mathcal{D}\phi\, \mathcal{D}\alpha\, \mathcal{D}\Omega\; e^{-\,S\,-\,\int_{\partial M} E\Omega[\phi\,+\,2E^{-1}\dot{\alpha}]} \tag{3.3'}$$

where we have introduced an auxiliary field Ω to exponentiate the delta functional. The functional measure for the Ω integration is also chosen to be invariant under translations of the Liouville field like (3.8).

4. The Jacobian

We introduce a parameter $\lambda \in [0,1]$ to define a family of metrics

$$g_{ab}(z,\lambda) \;\equiv\; e^{2\lambda\phi} g_{ab}(z)$$

interpolating between the fiducial metric and the Weyl-transformed one. The corresponding interpolating einbeins on the boundary may be defined similarly,

$$E(\lambda) \;\equiv\; e^{\lambda\phi} E\ .$$

The Jacobian $exp - J$ in a transition to a translationally invariant measure can be written down formally as

$$e^{-J} \;=\; Det^{\frac{1}{2}}[e^{\phi(z_1)\,+\,\phi(z_2)}\delta_{g(\lambda)}(z_1,z_2)]\ , \tag{4.1}$$

where, $\delta_{g(\lambda)}(z_1,z_2)$ is the covariant delta function appropriate to the interpolating metric. For infinitesimal values of λ, one obtains

$$\delta J \;=\; -\,Tr_{g(\lambda)}[\phi\delta\lambda]\ \ ; \tag{4.2}$$

the trace on the rhs is infinite and needs to be regularized. We use the heat kernel regulator [7,8] and rewrite (4.2) as

$$J_{reg} \;=\; -\,lim_{\epsilon\to 0}\int_0^1 d\lambda\; Tr_{g(\lambda)}\ \left[\phi e^{-\epsilon\triangle_{g(\lambda)}}\right]\ \ , \tag{4.3}$$

where $\triangle_{g(\lambda)}$ is the covariant Laplacian corresponding to the interpolating metric.

The trace in the integrand on the rhs may be evaluated using heat kernel expansion formulas derived in ref. [1], yielding

$$
\begin{aligned}
J_{reg} \ &= \ - \ lim_{\epsilon \to 0} \int_0^1 d\lambda \ \{ \frac{1}{4\pi\epsilon} \int_M \sqrt{g(\lambda)} \phi(z) \ - \ \frac{1}{8\sqrt{\pi\epsilon}} \int_{\partial M} E(\lambda) \phi(z_0(s)) \\
&+ \ \frac{1}{12\pi} \left[\tfrac{1}{2} \int_M \sqrt{g(\lambda)} R(g(\lambda)) \phi(z) \ + \ \int_{\partial M} E(\lambda) k(E(\lambda)) \phi(z_0) \right] \\
&- \ \frac{1}{8\pi} \int_{\partial M} E(\lambda) n^a(\lambda) \partial_a \phi(z_0(s)) \ + \ \mathcal{O}(\epsilon^{\frac{1}{2}}) \} \ .
\end{aligned} \tag{4.4}
$$

To perform the integral over λ, observe that for a Weyl transformation $g_{ab} \ \to \ e^{2\sigma} g_{ab}$, one has the relations

$$
\begin{aligned}
R(e^{2\sigma} g) \ &= \ e^{-2\sigma} [R(g) \ + \ 2\Delta_g \sigma] \\
k(e^{\sigma} E) \ &= \ e^{-\sigma} [k(E) \ - \ n^a \partial_a \sigma] \ .
\end{aligned} \tag{4.5}
$$

Using these and performing the resulting quadratures we get

$$
\begin{aligned}
J_{reg} \ &= \ - \ lim_{\epsilon \to 0} \{ \frac{1}{8\pi} \int_M \sqrt{g} (e^{2\phi} \ - \ 1) \ - \ \frac{1}{8\sqrt{\pi\epsilon}} \int_{\partial M} E(e^{\phi} \ - \ 1) \\
&+ \ \frac{1}{12\pi} \left[\tfrac{1}{2} \int_M \sqrt{g} \ (R \ + \ \Delta_g \phi) \ \phi \ + \ \int_{\partial M} E \ (k \ - \ \tfrac{1}{2} \partial_n \phi) \ \phi \right] \\
&- \ \frac{1}{8\pi} \int_{\partial M} E \ \partial_n \phi \} \ .
\end{aligned} \tag{4.6}
$$

Further simplification is obtained by use of Stokes' Theorem which provides the useful relation

$$
\int_M \sqrt{g} \ \phi \Delta_g \phi \ = \ \int_{\partial M} E \ \phi \partial_n \phi \ + \ \int_M \sqrt{g} g^{ab} \partial_a \phi \partial_b \phi \ \ , \tag{4.7}
$$

yielding finally

$$
\begin{aligned}
J_{reg} \ &= \ - \ lim_{\epsilon \to 0} \left[\frac{1}{8\pi\epsilon} \int_M \sqrt{g} \ (e^{2\phi} \ - \ 1) \ - \ \frac{1}{8\sqrt{\pi\epsilon}} \int_{\partial M} E \ (e^{\phi} \ - \ 1) \right] \\
&+ \ \frac{1}{12\pi} \left[\tfrac{1}{2} \int_M \sqrt{g} \ (g^{ab} \partial_a \phi \partial_b \phi \ + \ R(g) \phi) \ + \ \int_{\partial M} E \ k\phi \right] \ + \ \frac{1}{8\pi} \int_{\partial M} E \ \partial_n \phi \ .
\end{aligned} \tag{4.8}
$$

The regularized Jacobian is then inserted into the partition function (3.3') which takes the form

$$
Z(g, E) \ = \ \int \mathcal{D}\Omega \ \mathcal{D}\alpha \ \mathcal{D}_t \phi \ e^{-\{ S_{eff} \ + \ \int_{\partial M} E \ \Omega(\phi \ + \ 2E^{-1} \dot{\alpha}) \}} \ \ , \tag{4.9}
$$

where,

$$
\begin{aligned}
S_{eff} \; &= \; S(g,E,0) \; + \; \frac{3Q_0^2 \; - \; 1}{12\pi} \int_M \sqrt{g} \; (g^{ab}\partial_a\phi\partial_b\phi \; + \; R(g)\phi) \\
&+ \; \frac{3Q_0^2 \; - \; 1}{12\pi} \int_{\partial M} E \; k\phi \; + \; \frac{1}{4\pi}(\,\mu_0^2 \; - \; \frac{1}{2\epsilon}\,) \int_M \sqrt{g} e^{2\phi} \\
&+ \; \frac{1}{8\sqrt{\pi}}(\,\lambda_0 \; - \; \frac{1}{\sqrt{\epsilon}}) \int_{\partial M} E e^{\phi} \; + \; \frac{1}{8\pi}(\,\rho_0 \; - \; 1\,) \int_{\partial M} E \; \partial_n \phi
\end{aligned}
\tag{4.10}
$$

Before we go on to discuss the renormalization of the partition function in the above equations, we note that the integrals over $\alpha(s)$ and $\Omega(s)$ can be performed in that order. Since S_{eff} is independent of both these functions, our concern is only with the second term in the exponent in (4.9). The integral over α yields

$$
Z(g,E) \;=\; \int \mathcal{D}\phi e^{-S_{eff}} \int \mathcal{D}\Omega e^{-\int_{\partial M} E\Omega\phi} \, \delta[\dot{\Omega}(s)] \;\; , \tag{4.11}
$$

where, a partial integration has been performed first to flip the s-derivative on Ω. Since $\dot{\Omega} \; = \; 0 \; \leftrightarrow \; \Omega(s) \; = \; \Omega(0) \; \equiv \; \Omega_0$, the integral over Ω becomes trivial, producing the result

$$
Z(g,E) \;=\; \int \mathcal{D}_t\phi e^{-\left[S_{eff} \, + \, \Omega_0 \int_{\partial M} E \; \phi(z_0(s))\right]} \;\; . \tag{4.12}
$$

Consequently, the boundary value of the Liouville field $\phi(z_0(s))$ is rendered unconstrained by this manipulation, albeit at the expense of introducing a new boundary term proportional to ϕ into the action. The constant Ω_0 is arbitrary.

5. Renormalization and Anomalous Dimensions

It is clear from eq.s (4.9-10) that the incorporation of the Jacobian produces terms in the 'effective' action S_{eff} which are renormalizable. The simplest of the renormalizations is the finite renormalization of the coupling constant $Q_0 \; \equiv \; [\frac{1}{3}(26 - c_m)]^{\frac{1}{2}}$; from (2.18) we see that the renormalized coupling constant Q is given by

$$
3Q^2 \;\; = \;\; 3Q_0^2 \; - \; 1 \; = \; 25 \; - \; c_m \;\; , \tag{5.1}
$$

which is the well-known value of the 'background charge' obtained for the case without the boundary [3,7,8]. Here, as in [7,8], it emerges directly from the computation of the Jacobian, using a Weyl-noninvariant regularization. Observe that the boundary term linear in ϕ has the same parameter in front provided we rescale the constant Ω_0 appropriately. Thus, boundary effects do not alter the bulk renormalization of the charge Q, although the earlier interpretation of this constant as a background charge [3] at infinity for a free conformal field theory is perhaps no longer valid.

Observe also that the above result emerged on the assumption that the measures for the integration over $\alpha(s)$ and $\Omega(s)$ were invariant under shifts of the Liouville field. If, instead, we had employed a Weyl-invariant measure, and appropriately regularized the resulting Jacobian as above, our conclusions might have been different. We hope to report on this elsewhere.

The renormalization of the bulk cosmological constant can be performed as usual : one writes $\mu_0^2 = \mu^2 + \delta\mu^2$ and chooses the counterterm $\delta\mu^2$ to eliminate the divergent term. In ref. [3], the choice of the counterterm is made to eliminate the *renormalized* cosmological term altogether, but this will not be necessary for our purpose. The divergent piece of the 'boundary cosmological term', namely $\int_{\partial M} Ee^{\gamma\phi}$ can be similarly eliminated by defining $\lambda_0 = \lambda + \delta\lambda$, as also the finite renormalization of the normal derivative term. We must also include a wave function renormalization for the ϕ field involving a parameter γ which will determine anomalous dimensions of certain operators. We scale the Liouville field in such a way that this wave function renormalization parameter γ resides only in the cosmological constant terms. Finally, rescaling the (renormalized) ϕ field as $\phi \rightarrow \phi/Q$, we can retrieve the canonical form for the kinetic energy term for the Liouville field. Thus, in sum, we get the 'conformal' piece of the Polyakov partition function for genus zero,

$$\begin{aligned} Z(g,E,\Omega_0) &= \int \mathcal{D}\phi exp \; - \; \{ \frac{1}{8\pi}\int_M \sqrt{g}\,[\,g^{ab}\partial_a\phi\partial_b\phi \; + \; Q\,R(g)\,\phi\,] \\ &+ \; \frac{Q}{8\pi}\int_{\partial M} E\,(k \; + \; \Omega_0)\phi \; + \; \frac{\mu^2}{4\pi}\int_M \sqrt{g}\,e^{2\gamma\phi} \; + \; \frac{\lambda}{8\sqrt{\pi}}\int_{\partial M} Ee^{\gamma\phi} \\ &- \; \frac{\rho}{8\pi\gamma}\int_{\partial M} E\partial_n\phi\,\} \; . \end{aligned} \tag{5.2}$$

We still have to determine the parameter γ; since the same parameter characterizes both the bulk and the boundary cosmological terms, one expects γ also to be identical to the bulk value [3], namely given by the quadratic equation

$$2\gamma^2 \; - \; Q\gamma \; + \; 1 \;\; = \;\; 0\;. \tag{5.3}$$

This follows from the fact that the energy momentum tensor has the usual bulk contribution

$$\begin{aligned} T^M_{ab} \; &= \; \tfrac{1}{2}[\,\partial_a\phi\partial_b\phi \; - \; \tfrac{1}{2}\eta_{ab}(\partial\phi)^2 \\ &+ \; Q(\partial_a\partial_b\phi \; - \; \tfrac{1}{2}\eta_{ab}\Box\phi) \; + \; \mu^2\eta_{ab}e^{2\gamma\phi}\,]\;, \end{aligned} \tag{5.4}$$

so that the operator product expansion with the operator $e^{2\gamma\phi}$ produces exactly the same singularities as in the bulk analysis [3,6,7]. Eq. (5.3) arises if this operator is required to be a primary field of the Liouville-matter CFT with conformal dimension (1,1).[4]

The situation is less clear with regard to the boundary terms. First of all, the straightforward calculation of the energy momentum tensor from the Liouville action in (4.10) has a contribution from the boundary given by,

$$T_{ab}^{\partial M} = \{ Q[(\Omega_0 + k_0) - n^c\partial_c]\phi + \lambda e^{\gamma\phi} \} t_a t_b \tag{5.5}$$

where of course, $\phi = \phi(z_0(s))$. Thus the total energy momentum tensor is the sum of the bulk and boundary tensors given by eq. (5.4,5). Now, while the trace of the bulk tensor vanishes for $\mu = 0$ independently of Q, for the boundary contribution to be traceless one needs to satisfy the condition

$$n^a\partial_a\phi = (k_0 + \Omega_0)\phi + \frac{\lambda}{Q}e^{\gamma\phi} , \tag{5.6}$$

so that even for $\lambda = 0$ one has a nontrivial condition for the ϕ field. This condition in fact ensures that the entire boundary contribution to the energy momentum tensor vanishes. It follows that the boundary contribution violates the conformal symmetry of the ϕ field action even for vanishing cosmological terms.

The full import of these boundary terms is yet to be worked out. Essentially they generate reparametrizations of the affine parameter parametrizing ∂M. But this breaks the 2d conformal invariance of the theory. It is not clear that if we had started with Neumann type boundary conditions and retraced the steps we have followed so far, the boundary contribution would have vanished. This is particularly true for terms depending on the extrinsic curvature k of ∂M. Recall that k appears in the formula for the Euler characteristic of the two-fold, and is quite independent of the boundary condition on ϕ. Clearly therefore, much more remains to be investigated before we understand these boundary terms.

6. Conclusion

The innocuous nature of the boundary effects discerned by us may be traced to our employment of translationally invariant integration measures for integrations over boundary reparametrizations $\alpha(s)$ and the auxiliary field $\Omega(s)$. Thus the final Liouville action decomposes into mutually independent contributions from the bulk and the boundary with the latter having almost no effect on the renormalizations of the parameters of the former. The same is true for the energy momentum tensors.

[4]We are assuming of course that no subtlety arises when free field OPEs are used

The only non-trivial feature that emerges is that, after 'deconstraining' the boundary value of ϕ, 2d conformal invariance of the theory (for vanishing cosmological terms) is broken by the boundary contributions. Further constraints must be imposed on the Liouville field on the boundary if these terms are to be eliminated.

It is interesting, although perhaps not surprising, to note that the condition (5.6) above for the vanishing of $T_{ab}^{\theta M}$ reduces to the Neumann boundary conditions used in ref.[5] to analyze boundary operators in 2d quantum gravity, provided of course we set $k_0 + \Omega_0 = 0$. In this case, with the cosmological terms renormalized to zero a la' ref. [3], the use of the free field OPEs in [5] to calculate the so-called boundary dimension of boundary operators is perfectly in order. The question that remains however, is that if the constraint above were to be imposed from the outset instead of those of ref. [1], i.e., eq.(2.7), will the boundary terms in the energy momentum tensor vanish. This is currently under investigation, as also is the issue of using a Weyl-invariant measure for integrations over $\alpha(s)$ and $\Omega(s)$.

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